J. Inf. Rec. Mater. **16** (1988) 5, 377–385

A simple approximation for the characteristic curve of photographic emulsions and the evaluation of the parameters. Part II.

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Summary

Some usual approximation functions for the analytical representation of the characteristic curve are presented, of which in particular a simple one is discussed, that warrants a high accuracy in the linear range and the toe. A method to determine the function parameters is given, and the correlation with Gerth's step-order number is investigated.

Резюме

Обсуждаюгся некоторые часто используемые функции приблежения кривых почернения. Представляется и рассматривается простая функция с достаточной точностью без учёта области насыщения и даётся метод определения параметров функции. Также обсуждается взаимосвязь n используемого Гертом параметра роста $H_Г$.

параметра

Zusammenfassung

Zur analytischen Darstellung der Schwärzungskurve werden einige gebräuchliche Approximationsfunktionen vorgestellt und insbesondere eine einfache Funktion diskutiert, die eine hohe Genauigkeit im linearen Bereich und im Durchhang gewährleistet. Eine Methode zur Bestimmung der Funktionsparameter wird angegeben und der Zusammenhang zur Gerthschen Keimordnungszahl untersucht.

1. Introduction

Depending on the special problem of application differently approximated or interpolated functions are customary used the photographic characteristic curve representing. Гi/y
Hence it is not to be wondered that there are a lot of different functions and, likewise, related methods for determination and evaluation of parameters, contained in the formula.

The number of nodes and the accuracy required are of importance for the selection of a suitable function. The efforts for calculation of the function and of the parameters have become unessential by progress in computing techniques nowadays. In our case, however, we need a simply calculable function restricting the expense of computing time as much as possible. On the other hand, the number of parameters must be small if there is available a small set of nodes (say 6 or 8) only. An accuracy of 3 per cent has to be secured, apart from the range of the shoulder. It would be desirable to have at hand !!

the explicite values for the characteristic curve as well as for its gradient at the point of inflection. In principle, the density S depends on the effective exposure energy H in form of $S = S(H^n)$. The exponent n here is a approximating parameter, which needs not to be identical with GERTH's step number n_g (see part I). The density is to be calculated by $T = T_0 \cdot \exp(-S)$, where T is the transmission. The fog has to be subtracted from all density values so that $S(0) = 0$. The effective exposure energy comes out with the exposure intensity I, the exposure time t, and the SCHWARZSCHILD exponent p as $H = I \cdot t^p$. Occasionally, the logarithm $X = \ln H$ of the effective exposure energy H is used for the representation of the approximated density function. To construct approximation functions for the characteristic curve, B. R. FRIEDEN (1985) has given an other method, superimposing some orthogonal functions.

2. Simple approximation functions

At first we consider some commonly used approximation functions. Often a polynomial optimization $S = P_\nu(H)$ is used for the approximation of curves on the base of a given set of nodes. As general as this ansatz may be, so it has some unavoidable disadvantages. In order to achieve the required accuracy, a satisfactorily high degree ν of the polynomial has to be chosen. Thus the deviations of the nodes H_i, S_i, $i = 1, \ldots, N$ might lead to se same senless function values. Thus, for instance, more flection points may occur than that are expected. The value of ν has to be fitted to the set of nodes with given accuracy. In practice the required accuracy can usually not be achieved if only few nodes ($N = 6$ or 8) are available.

Better results can be obtained by means of spline-interpolation, however, for $N = 6$ the accuracy is not sufficient, too.

Using the results of the characteristic functions given in part I, and extending the polynomial in the denominator to higher powers, we find

$$S = S_\infty \cdot \left[1 - \frac{1}{1 + a_1 H^n + a_2 H^{2n} + a_3 H^{3n}}\right]. \tag{1}$$

As a consequence $S \simeq S_\infty \cdot a_1 \cdot H^n$ for $S \ll S_\infty$. Even a representation of the shoulder range can be achieved by means of this function. If it is not possible to measure the saturation density S directly, this value can only be obtained by means of approximation methods. The approximation procedure leads to unstable values for S_∞ and for the other parameters. The reproduced curves are very different from each other, which may be produced already by a small deviation of only one node. The flection point has to be localized by approximation. Likewise the inversion can be carried out by approximation only. However, this entails essential difficulties for values $S > 0.9 \cdot S_\infty$.

The parameters a_1, a_2, a_3 are to be determined by some preliminarily given values of the characteristic curve, e.g. the inflexion point and the saturation density. With $H = \exp(X)$, $\sigma = S_\infty/(S_\infty - S)$ and $y = (\mathrm{d}S/\mathrm{d}X)/nS_\infty$ it follows that

$$\begin{aligned}
&a_1 \cdot \exp(nX) + a_2 \cdot \exp(2nX) + a_{\cancel{2}}^{3} \cdot \exp(3nX) = \sigma - 1, \\
&a_1 \cdot \exp(nX) + 2a_2 \cdot \exp(2nX) + 3a_3 \cdot \exp(3nX) = y \cdot \sigma^2, \\
&a_1 \cdot \exp(nX) + 4a_2 \cdot \exp(2nX) + 9a_3 \cdot \exp(3nX) = 2y^2\sigma^3 + \sigma^2(\mathrm{d}^2S/\mathrm{d}X^2)/S_\infty.
\end{aligned} \tag{2}$$

The coefficients a_i may be derived from the values S_w, X_w, γ_{xw}, (σ_w, y_w) of the flection point $(\mathrm{d}^2S/\mathrm{d}X^2 = 0)$ as

$$\begin{aligned}
a_1 &= \exp(-nX_w) \cdot [3(\sigma_w - 1) + y_w\sigma_w^2(y_w\sigma_w - 5/2)], \\
a_2 &= \exp(-2nX_w) \cdot [-3(\sigma_w - 1) - 2y_w\sigma_w^2(y_w\sigma_w - 2)], \\
a_3 &= \exp(-3nX_w) \cdot [(\sigma_w - 1) + y_w\sigma_w^2(y_w\sigma_w - 3/2)].
\end{aligned} \tag{3}$$

Another phenomenological ansatz

$$\frac{\mathrm{d}S}{\mathrm{d}H} = \alpha \cdot n \cdot H^{n-1} \cdot \frac{(S_\infty - S)^{m+1}}{(S_\infty - S_0)^m}, \qquad n > 1, \qquad m \geqq 0 \tag{4}$$

yields the characteristic function

$$S/S_\infty = 1 - \frac{(1 - S_0/S_\infty)}{(1 + \alpha m H^n)^{1/m}}, \qquad \alpha m H^n = \left[\frac{S_\infty - S_0}{S_\infty - S}\right]^m - 1, \qquad m \neq 0, \tag{5}$$

with the asymptotical behaviour $S \simeq S_0 + (S_\infty - S_0) \cdot \alpha \cdot H^n$ for $\alpha m H^n \ll 1$. For the point of inflection, we have

$$\alpha H_\mathrm{w}{}^n = \frac{n-1}{m+n}, \qquad \frac{S_\infty - S_\mathrm{wH}}{S_\infty - S_0} = \left[\frac{n+m}{n(m+1)}\right]^{1/m}, \tag{6}$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}H}\right|_\mathrm{w} = \alpha^{1/n} \cdot (S_\infty - S_0) \cdot \left[\frac{n-1}{m+n}\right]^{1-1/n} \cdot \left[\frac{n+m}{n(m+1)}\right]^{1+1/m}. \tag{7}$$

In case of $m = 0$, the ansatz (4) gives

$$S/S_\infty = 1 - (1 - S_0/S_\infty) \exp(-\alpha H^n), \qquad m = 0, \tag{8}$$

with the relations for the flection point

$$\alpha H_\mathrm{w}{}^n = \frac{n-1}{n}, \qquad \frac{S_\infty - S_\mathrm{wH}}{S_\infty - S_0} = \exp(1/n - 1), \tag{9}$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}H}\right|_\mathrm{w} = \alpha^{1/n} \cdot n(S_\infty - S_0) \cdot \left[\frac{n-1}{\mathrm{e} \cdot n}\right]^{1-1/n}. \tag{10}$$

The range of S is in all cases $S_0 < S < S_\infty$. The parameters α and n characterize the 'toe', and the remaining parameter m defines the position and the gradient of the flection point, depending on each other. This representation of the characteristic curve proves to be advantageous for the explicit representation of the values γ_w and H_w, S_wH; it excludes nonresonable curves. But it demands ~~an~~ considerable effort to determine the parameters.

Approximation functions of the form

$$\lg H = a_1 + a_2 \cdot S + a_3 \cdot \ln\left(\exp(bS^{c_1}) - 1\right) + a_4 \cdot \exp(bS^{c_2}) \tag{11}$$

have gained a wide spread (Becker 1979, see further references there). It has been pointed out that a sufficient approximation can often be achieved also without the last term of equation (11). The determination of the parameters of the function (11) is more complicated and takes more effort of calculation than our formula presented below and, apart from this, it does not exist any explicit representation of the characteristic values.

An other version of the characteristic function, comes out of the theory of the photographic process regarded to thin layers and uniform grain size, as an integrallogarithm function. A series expansion yields

$$S(H) = S_0 + S_\mathrm{m}(H/H_\mathrm{m})^n \cdot \exp\left\{n\left[-\frac{(H/H_\mathrm{m})^1}{1 \cdot 1!} + \frac{(H/H_\mathrm{m})^2}{2 \cdot 2!} - \frac{(H/H_\mathrm{m})^3}{3 \cdot 3!} + \cdots\right]\right\}. \tag{12}$$

This relation suggests the ansatz

$$S(H) = S_0 + S_{\mathrm{m}}(H/H_{\mathrm{m}})^n \exp\left(-nP\nu(H/H_{\mathrm{m}})\right) \tag{13}$$

with the polynomial P_ν, $P_\nu(0) = 0$. A polynomial of the order $\nu = 3$ already renders a sufficient fitting. However, for our problems we need the inverse characteristic function $H = H(S)$, the socalled *exposure function,* which will be discussed in the following section.

3. Representation of a characteristic curve by approximation of an exposure function

The ansatz for the approximation of an exposure function

$$H^n(S) = \frac{(S - S_0)}{S_{\mathrm{m}}} \exp\{P_\nu(S - S_0)\} \tag{14}$$

renders similarly good results as (11) and (13) do, except for the upper part of the shoulder and the saturation. The exponent n is assumed to be greater than zero, $n > 0$. The accuracy required is achieved by a polynomial of the order $\nu = 2$. Hereby, an explicit representation of the properties characterizing the curve is given, offering a criterion for reasonable curves. Without any restriction, by putting $S_0 = 0$, we find our fundamental formula

$$\boxed{H^n = \frac{S}{S_{\mathrm{m}}} \exp[b_1 S + b_2 S^2] = S \cdot \exp[b_0 + b_1 S + b_2 S^2]} \tag{15}$$

with $b_0 = -\ln S_{\mathrm{m}}$. We have $S \simeq S_{\mathrm{m}} H^n$ for $H^n \ll 1/S_{\mathrm{m}}$. Usually, S is represented as a function of X, with

$$\boxed{X = \ln H = \frac{1}{n}(\ln S + b_0 + b_1 S + b_2 S^2)} \tag{16}$$

yielding $S \simeq S_{\mathrm{m}} \exp(nX) = \exp(nX - b_0)$ for $nX \ll 1$. The gradient of the curve is

$$\gamma_{\mathrm{x}} = \frac{\mathrm{d}S}{\mathrm{d}X} = \frac{nS}{1 + b_1 S + 2b_2 S^2}. \tag{17}$$

The point of inflection has the coordinates

$$S_{\mathrm{wx}} = \frac{1}{\sqrt{2b_2}}, \qquad X_{\mathrm{w}} = \frac{1}{n}\left[b_0 + \frac{1}{2}(1 - \ln 2b_2) + \frac{b_1}{\sqrt{2b_2}}\right], \qquad b_2 > 0 \tag{18}$$

and the gradient is given by

$$\gamma_{\mathrm{wx}} = \frac{n}{b_1 + 2\sqrt{2b_2}}. \tag{19}$$

The condition that γ_{wx} is a finite positive value leads to

Γb_1

$$b_1 > -2\sqrt{2b_2}. \tag{20}$$

If, contrary to this, $b_1 < -2\sqrt{2b_2}$, we have $\gamma_{wx} < 0$, and the gradient γ_x is indefinite at both density values S_1, S_2 with

$$S_{1,2} = \frac{(-b_1)}{4b_2}\left\{1 \pm \sqrt{1 - \frac{8b_2}{b_1{}^2}}\right\}. \tag{21}$$

In this case either the range $0 < S < S_2$ (the 'toe') or the range $S > S_1$ (the 'shoulder') is appropriate for approximation. If no point of inflection exists, e.g. $b_2 < 0$, the gradient $\gamma_x(S_E)$ is indefinite for the particular density value S_E,

$$S_E = \frac{b_1}{4(-b_2)}\left\{1 \pm \sqrt{1 + \frac{8(-b_2)}{b_1{}^2}}\right\}, \quad \begin{cases} +, & \text{for} \quad b_1 > 0, \\ -, & \text{for} \quad b_1 < 0. \end{cases} \tag{22}$$

Thus, the approximation is applicable in the range $0 \leqq S < S_E$ (toe), also. Examples of these three cases are shown in Fig. 1. Except for the representation of the saturation we economize one parameter. The parameter S_m (or b_0) contains the scales of the exposure energy H and that of the density S (one parameter, because we renounced the saturation density); the parameter n is determined by the form of the toe.[1])
The parameter $b_2 = 1/2S_{wx}^2$ is determined by the density at the point of inflection, whereas the gradient γ_{wx} regulate the parameter $b_1 = n/\gamma_{wx} - 2/S_{wx}$. L/ rules

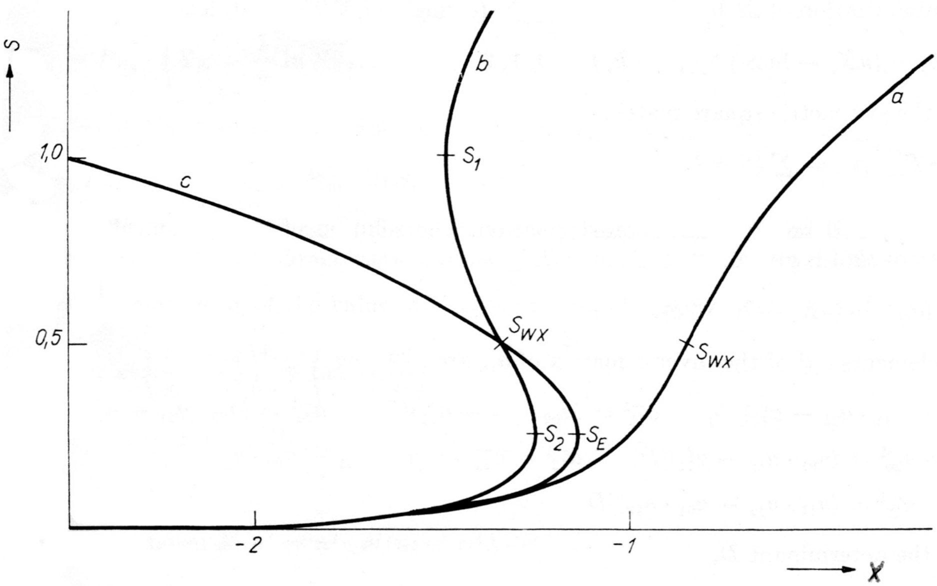

Fig. 1. The representation of the approximation function (16); a) be valid for a density function in the case $b_2 > 0$ and $b_1 > -2\sqrt{2b_2}$; b) the case $b_2 > 0$ but $b_1 < -2\sqrt{2b_2}$, limited to either $0 < S < S_2$ (toe) or $S > S_1$ (shoulder); c) the case $b_2 < 0$ — without flection point — suitable only for $0 \leqq S < S_E$ (toe).

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[1]) $$\frac{d^2S}{dX^2} = \frac{n^2S}{(1 + b_1S + 2b_2S^2)^3}(1 - b_2S^2), \quad \begin{cases} \xrightarrow{S\to 0} n^2S, \\ \xrightarrow{S\to\infty} -n^2/(8b_2{}^2S^3). \end{cases}$$

For the curvature K of the curve we have
$K \xrightarrow{S\to 0} n^2S, \quad K \xrightarrow{S\to\infty} -n^2/(8b_2{}^2S^3), \quad K(S_{WX}) = 0.$

4. Determination of the parameters

The approximation function may be written as

$$S_{ij} \cdot b_j = n \cdot X_i - \ln S_i\,, \qquad i = 1, \ldots, N\,, \qquad j = 0, 1, 2\,, \tag{23}$$

where the index i labels the nodes X_i, S_i $(X_i = \ln H_i)$. The coefficient matrix S_{ij} of the set of algebraic equations is

$$S_{ij} = (S_i)^j\,, \qquad j = 0, 1, 2\,. \tag{24}$$

The system is overdetermined $(N > 3)$ and incompatible, i.e., the nodes X_i, S_i do not coincide with the curve of the approximation function (15). Without loss of generality, the nodes S_i may be taken as accurate values, whereas the values X_i deviate from the values $\overline{X}_i$, arranged exactly on the curve, by $\delta\overline{X}_i$. Therefore, the matrix S_{ij} is fixed. The deviations $\delta\overline{X}_i$ are adopted as randomly distributed according to a GAUSSIAN function. The best approximation is reached for the determination of the parameters b_0, b_1, b_2 and n by minimizing of the square norm $M = \sum (\delta X_i)^2$ of the deviations (the method of least squares). At first we assume the parameter n being provisionally known. For

$$n\delta\overline{X}_i = n(X_i - \overline{X}_i) = nX_i - \ln S_i - S_{ij} \cdot b_j \tag{25}$$

the minimization of M by the parameter b_k (owing to $\partial M/\partial b_k = 0$) leads to

$$a_{kj} \cdot b_j = (nX_i - \ln S_i)\, S_{ik}\,, \qquad k, j = 0, 1, 2\,, \tag{26}$$

with the symmetric square matrix

$$a_{kj} = S_{ik} \cdot S_{ij} = \sum_i (S_i)^{k+j}\,. \tag{27}$$

/the This is ~~be~~ well known GAUSS transformation. The solution of the system of algebraic equations (26) is given by

$$b_l = (a_{lk})^{-1} \cdot (nX_i - \ln S_i)\, S_{ik}\,. \tag{28}$$

The elements a_{lk}^{-1} of the inverse matrix of a_{lk} are

$$\begin{aligned}
&a_{00}^{-1} = (a_{11} \cdot a_{22} - a_{12}^2)/D, \qquad a_{11}^{-1} = (a_{00} \cdot a_{22} - a_{11}^2)/D, \qquad a_{22}^{-1} = (a_{00} \cdot a_{11} - a_{01}^2)/D,\\
&a_{02}^{-1} = a_{20}^{-1} = (a_{01} \cdot a_{12} - a_{11}^2)/D, \qquad a_{12}^{-1} = a_{21}^{-1} = (a_{01} \cdot a_{11} - a_{00} \cdot a_{12})/D\\
&a_{01}^{-1} = a_{10}^{-1} = (a_{11} \cdot a_{12} - a_{01} \cdot a_{22})/D
\end{aligned} \tag{29}$$

with the determinant D,

$$D = |a_{lk}| = a_{00} \cdot a_{11} \cdot a_{22} + 2a_{01} \cdot a_{11} \cdot a_{12} - a_{00} \cdot a_{12}^2 - a_{11}^2 - a_{22} \cdot a_{01}^2\,. \tag{30}$$

An identical result follows by minimization of

$$\overline{M}(n) = n^2 \cdot M(n) = \sum_m (n\delta\overline{X}_m)^2\,, \tag{31}$$

L δ with $n\delta\overline{X}_m = \ln (1 + \delta H_m/H_m)^n$, being the deviation of the right hand side term of (23). This term, in form of $nX_i - \ln S_i = n(\overline{X}_i + \cancel{l}\overline{X}_i) - \ln S_i$, with $n\overline{X}_i - \ln S_i$ being as the reproduced part. The deviation $n\delta\overline{X}_i$ may be described by some kind of projection.

The projection tensors

$$\boldsymbol{L} = \{L_{\mathrm{im}}\}, \qquad \boldsymbol{P} = \boldsymbol{I} - \boldsymbol{L} = \{P_{\mathrm{im}}\} = \{\delta_{\mathrm{im}} - L_{\mathrm{im}}\} \tag{32}$$

are defined by

$$\boldsymbol{L}^2 = \boldsymbol{L}, \qquad \boldsymbol{P}^2 = \boldsymbol{P}, \qquad \boldsymbol{L} \cdot \boldsymbol{P} = 0 \tag{33}$$

and the condition for symmetry, $L_{\mathrm{im}} = L_{\mathrm{mi}}$, $P_{\mathrm{im}} = P_{\mathrm{mi}}$. Using the equations (23) and (28), we have

$$L_{\mathrm{mi}} = S_{\mathrm{ml}} \cdot (a_{\mathrm{lk}})^{-1} \cdot S^T_{\mathrm{ki}}, \qquad i, m = 1, \ldots, N, \qquad k, l = 0, 1, 2, \tag{34}$$

and

$$L_{\mathrm{mi}} \cdot S_{\mathrm{in}} = S_{\mathrm{mn}}, \qquad P_{\mathrm{mi}} S_{\mathrm{in}} = 0. \tag{35}$$

The reproduced values $\overline{X}_{\mathrm{i}}$, coinciding with the approximation function, are

$$\overline{X}_{\mathrm{i}} = L_{\mathrm{im}} \cdot X_{\mathrm{m}} + \frac{1}{n} P_{\mathrm{im}} \cdot \ln S_{\mathrm{m}}, \tag{36}$$

consisting of two parts being orthogonal vectors. The deviations are

$$\delta\overline{X}_{\mathrm{i}} = P_{\mathrm{im}} \cdot \left(X_{\mathrm{m}} - \frac{1}{n} \ln S_{\mathrm{m}}\right). \tag{37}$$

We denote the P-projection of X_{m} and $\ln S_{\mathrm{m}}$ by x_{m} and s_{m},

$$x_{\mathrm{m}} = P_{\mathrm{mi}} \cdot X_{\mathrm{i}}, \qquad s_{\mathrm{m}} = P_{\mathrm{mi}} \cdot \ln S_{\mathrm{i}}; \tag{38}$$

x^2, s^2 are the square norms of them, and $m = x_{\mathrm{m}} \cdot s_{\mathrm{m}}$ is the scalar product of the two vectors. Since for all characteristic curves $\sum_m X_{\mathrm{m}} \cdot \ln S_{\mathrm{m}} > 0$, there must be $m > 0$. The square norm M of the values $\delta\overline{X}_{\mathrm{i}}$ is a function of the parameter n:

$$M(n) = s^2 \left(\frac{1}{n} - \frac{m}{s^2}\right)^2 + \left(x^2 - \frac{m^2}{s^2}\right), \tag{39}$$

having the minimum $M_{\min} = x^2 - m^2/s^2 \geqq 0$ at

$$n = n^+ = s^2/m. \tag{40}$$

The square norm $\overline{\boldsymbol{M}}(n) = n^2 \cdot M(n)$ of $n\delta\overline{X}_{\mathrm{i}}$ is

$$\overline{\boldsymbol{M}}(n) = x^2 \left(n - \frac{m}{x^2}\right)^2 + \left(s^2 - \frac{m^2}{x^2}\right), \tag{41}$$

with the minimum $\overline{\boldsymbol{M}}_{\min} = s^2 - m^2/x^2 \geqq 0$ at

$$n = n^* = m/x^2. \tag{42}$$

Since $m > 0$ and $s^2x^2 \geqq m^2$, it follows $n^+ - n^* = (s^2x^2 - m^2)/mx^2 \geqq 0$ (equality exists only in the case $s_{\mathrm{m}} = n \cdot x_{\mathrm{m}}$, i.e. $\delta\overline{X}_{\mathrm{m}} = 0$ for all $m = 1, \ldots, N$). The value $n = n^+$ provides the best fitting for the quantities $\delta\overline{X}_{\mathrm{m}} = \ln(1 + \delta H_{\mathrm{m}}/H_{\mathrm{m}})$; on the other hand,

$n = n^*$ gives the best fitting for the quantity $n\delta \overline{X}_m = \ln (1 + \delta \overline{H}_m/H_m)^n = \ln (\overline{H}_m/H_m)^n$, because, in the case $n = n^+$ there is valid

$$M_{\min} = M(n^+) < M(n^*) = \overline{M}(n^*)/n^{*2} = \overline{M}_{\min}/n^{*2}, \tag{43}$$

on the other hand, if $n = n^*$, we have

$$\overline{M}_{\min} = \overline{M}(n^*) < \overline{M}(n^+) = n^{+2} M(n^+) = n^{+2} M_{\min}\,; \tag{44}$$

(and it follows $n^{*2} < n^{+2}$). The question arises, which of the two quantities, either n^+ or n^*, does give the better fitting to the step-order number n_g introduced by GERTH (see part I).

5. Determination of the step-order number

If the approximation function (15) should be usable for determination of the step-order number n_g, then to reproduced the characteristic curves on principle accurately, this function has i.e., the approximation functions has to coincide numerically with the exposure function within the range. It has, therefore, to be assumed, that $\delta \overline{X}_i$ vanishes if X_i, S_i takes the exact values.
The nodes X_i, S_i are affected by the deviations δX_i, δS_i, again putting $\delta S_i = 0$ without loss of generality. The deviations, δX_i or δH_i, render the step-order number n_g to be incorrect by δn. Now we replace the values of the nodes X_i by $X_i + \delta X_i$, X_i being the 'true' values, and n by $n_g + \delta n$.
The supposition for the determination of the step-order number may be formulated as: if $\delta X_i = 0$, then $\delta \overline{X}_i = 0$ and $\delta n = 0$. We have, generally,

$$s_i = n_g \cdot x_i \tag{45}$$

and all the other relations derived from eq. (45), e.g. $s^2 = n_g{}^2 \cdot x^2$, or $m = n_g \cdot x^2$. The deviations δX_i change the curvature of the approximation function by

$$\delta \overline{X}_i = L_{im} \cdot \delta X_m\,, \tag{46}$$

whereby the deviation of the nodes from the approximation function is

$$\delta \overline{X}_i = \delta x_i = P_{im} \cdot \delta X_m\,. \tag{47}$$

The square norm M of the last mentioned deviations is

$$M(\delta n) = \delta(x^2) + 2x \cdot \delta x \left(\frac{\delta n}{n_g + \delta n}\right) + x^2 \cdot \left(\frac{\delta n}{n_g + \delta n}\right)^2, \tag{48}$$

having the minimum

$$M_{\min} = \delta(x^2) - x^2 \left(\frac{x \cdot \delta x}{x^2}\right)^2 \tag{49}$$

at the value $n = n^+$ (see (40)). For the deviation $\delta n^+ = n^+ - n_g$ determined in this manner, we have

$$\frac{\delta n^+}{n_g} = -\frac{x \cdot \delta x}{x^2 + x \cdot \delta x}\,. \tag{50}$$

The minimum of the square norm $\overline{\boldsymbol{M}}(\delta n)$ of the deviations $(n_g + \delta n)\,\delta\overline{X}_i$ can be found at $n^* = n_g + \delta n^*$, yielding

$$\frac{\delta n^*}{n_g} = -\frac{\delta(x^2) + x \cdot \delta x}{x^2 + 2x \cdot \delta x + \delta(x^2)}. \tag{51}$$

From the formulae (50), and (51), one can see that, as a rule, n^+ gives the better approximation to the step-order number n_g than n^* because $\delta n^+ \approx -x \cdot \delta x / x^2$ oscillate around zero for different sets of nodes, with $x \cdot \delta x$ being the mean value of the deviations δx_i weighted by x_i. The deviation δn^* is less than δn^+ by the positive definite value $\approx \delta(x^2)/x^2$, therefore, as a rule, δn^* would be less than zero. Consequently, n^* tends to too small values, which has been widely corroborated by applications and numerical experiments. It is no surprice that the minimizing of $n\delta\overline{X}_m$ gives smaller values for n then the minimizing of $\delta\overline{X}_m$.
If several characteristic curves have to be determined for uniform emulsions, we have a common value n_g. Than we can base the computation on a unique value for the parameter n. The simultaneous calculation of the different characteristic curves yields

$$n^+ = \sum_k (s^2)_k \Big/ \sum_k m_k, \quad n^* = \sum_k m_k \Big/ \sum_k (x^2)_k \tag{52}$$

where $(s^2)_k$, $(x^2)_k$, m_k are the corresponding values s^2, x^2, m of the k-th single curve. The values n^+, n^* determined in this manner are to be preferred to the mean values $\overline{n^+}$, $\overline{n^*}$.

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Received: October 27, 1987.

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